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Optimal Ranging Codes

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ABSTRACT

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This Report provides an analysis of a continuous, coded ranging scheme. By the use of a Boolean function, several "component" sequences are encoded into a transmitted signal. The receiver correlates the delayed return signal with different Boolean combinations of delayed replicas of the components to determine separately the time delay of each component sequence. From these delays, the total delay is computed.

By proper choice of encoding logic, number and type of components, and the decoding logics and procedure, the range can be found in a relatively short time. Optimal parameters of this ranging device are derived.

I. INTRODUCTION

A ranging system (Ref. 1 and 2) is a radar device which can transmit a coded signal continuously and receive the delayed return signal, also continuously. Such a system is feasible whenever it is possible to isolate the transmitter from the receiver by distance, terrain, sufficient doppler shift, rebroadcast at a different frequency from a transponder on the target, or a combination of these. The advantages of continuous operation include maximum average-to-peak power ratios, variable integration time, continuous range measurement and tracking, and extreme accuracy.

One feature that must be incorporated into such a system is quick and easy initial range determination. Continuous operation will often require quite long codes, if no range ambiguity is to exist, especially when the range is hundreds of millions of kilometers, as one might encounter ranging a planetary spacecraft.

In the unconstrained channel with white, additive Gaussian noise, it has long been recognized that the optimum receiver is a set of correlators, or filters matched to each possible (assumed discrete) time-shifted return of the transmitted code (Ref. 3). For a long code, this requires a prohibitive amount of receiver equipment; and with only one correlator, serial operation requires an extremely long time to determine the range.

When the amount of receiver equipment is limited, matched filtering is thus no longer the optimal detection scheme. A better scheme, as is shown here, is one which, by the use of a Boolean function, combines several "component" sequences to generate the transmitted signal; the receiver quickly acquires the phase of each component and computes the range from this. This method was first suggested by Golomb (Ref. 4), and an operational model, built by Easterling (Ref. 5), has had amazing success ranging the planet Venus (Ref. 6).

This Report presents a general method for treating Boolean functions of component sequences. The optimal logics, component sequences, and number of components can be found by using the method.

II. COMPONENT-CODED RANGING CODES

A. The Acquisition Ratio

Suppose that a signal $x(t)$, generated by modulating a carrier by a sequence $\alpha = \{\alpha_n\}$ having period p, is sent through a simple continuous channel with white, additive Gaussian noise of zero mean as shown in Fig. 1. The time series $y(t)$ presented to the receiver is

$$y(t) = x(t - \tau) + n(t)$$

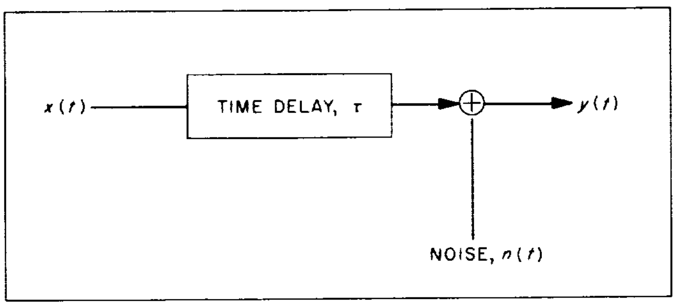

Fig. 1. The continuous channel

Here we assume no attenuation in the channel; we do this without loss in generality by assuming that the receiver is capable of amplifying $y(t)$ to recover any channel loss. The noise is, of course, also amplified, and this must be taken into account.

If t_0 represents the clock rate of the modulating sequence α, then the channel delay τ is, for some integer k,

$$\tau = kt_0 + \tau_0, \ (0 \leq \tau_0 < t_0)$$

Once τ_0 is found, the receiver "locks" this quantity out of the measurement on τ. We will assume, for the present, that such an *initial synchronization* or *clock lock* is in effect, and first consider cases with $\tau = kt_0$.

The optimum receiver to estimate k for the Gaussian channel is shown in Fig. 2. This receiver minimizes the error probability for a given detection time, or, equivalently, the detection time for a given probability of error. It consists of filters (or correlators) matched to each possible transmitted signal, and this, as indicated previously, generally requires a large amount of equipment. Sometimes we are limited to a certain amount of equipment or receiver complexity, and we must operate on the incoming signal accordingly.

For example, by using p correlators, we are able to estimate or "acquire" the time shift, or "phase" of the received signal with a certain probability of error after integrating for, say, T sec. This is the least T giving this probability of error. However, when limited to *one* correlator in the receiver, we must correlate the incoming signal serially against every phase shift of the incoming signal, which requires pT sec to achieve the same probability of error. There is thus a trade-off between receiver complexity and acquisition time:

$$T_{\text{acq}} = \frac{\text{time for a one-correlation receiver to acquire } \alpha}{\text{number of correlators in receiver}}$$

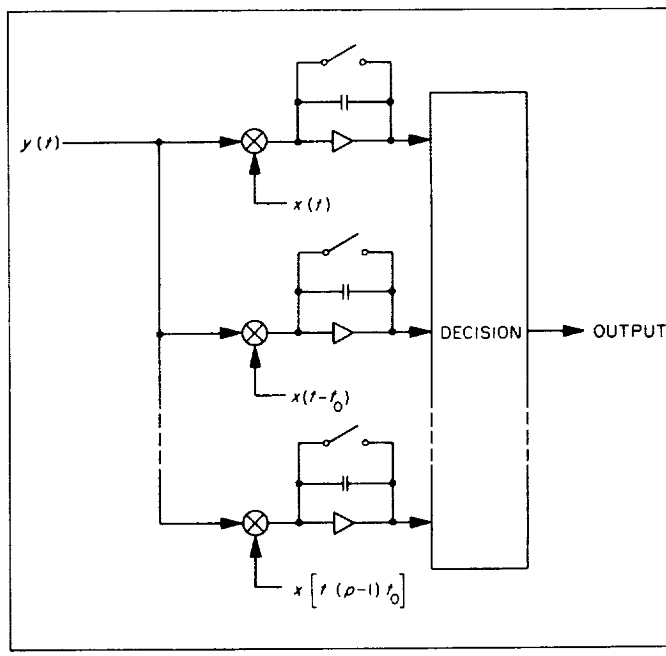

Fig. 2. The optimum receiver for the white-noise Gaussian channel

Now, as an alternative, suppose our scheme is to cross-correlate α against several locally generated sequences, say $\gamma_1, \gamma_2, \cdots, \gamma_n$. The cross-correlation function $C_{\alpha\gamma_i}(m)$ repeats itself cyclically with period

$$v_i = (p, u_i)$$

if γ_i has period u_i. That is, $C_{\alpha\gamma_i}(m + v_i) = C_{\alpha\gamma_i}(m)$.

Knowing the vector $\mathbf{m} = (m_1, m_2, \cdots, m_n)$ containing the delays m_i (reduced modulo v_i) at which each of the $C_{\alpha\gamma_i}(m)$ is a maximum, we must be able to decide the most probable value of k uniquely. The number of different vectors encountered must thus be greater than the number of phases of α, so

$$p \leq [v_1, v_2, \cdots, v_n]$$

Second, the period u_i of γ_i cannot be relatively prime to p, for if it were, $C_{\alpha\gamma_i}(m)$ would be the same for all m [because $v_i = (u_i, p)$]. Every v_i, therefore, divides p, and hence

$$p \geq [v_1, v_2, \cdots, v_n]$$

These last two inequalities indicate that

$$p = [v_1, v_2, \cdots, v_n]$$

With one integrator observing T' sec per step, the time required serially to perform all correlations of α with the γ_i, phase-by-phase and sequence-by-sequence, is $(v_1 + v_2 + \cdots + v_n)T'$. We choose T' sufficiently long that the confidence limits in this scheme are the same as the previous ones using integration time T. The *acquisition ratio,* defined as

$$(T'_{\text{acq}})/(T_{\text{acq}}) = [(v_1 + v_2 + \cdots + v_n)T']/(pT)$$

represents the relative saving, if any, between the two schemes, each with the same specified number of integrators.

If it were possible to pick γ_i, n, and T' in such a way that the ratio is less than unity, the alternate scheme would prove a more desirable receiver in that for a *given receiver complexity and error probability,* the *total time to acquire is less* in the second method. We will not only show that this is possible, but we will also give a way by which a great saving can be achieved.

B. Correlation Time as a Function of Distinguishability

We now wish to compare the integration time T required to give a constant probability of error as a function of correlation separation. Suppose a unit-power signal $x(t)$ is transmitted, $y(t) = x(t - m) + n(t)$ is received, and the receiver correlates $y(t)$ against a unit-power waveform $z(t)$ for a time T. The output $\Lambda(m, T)$ of the integrator is then

$$\Lambda(m, T) = \int_0^T y(t)\, z(t)\, dt$$

$$= \int_0^T x(t - m)\, z(t)\, dt + \int_0^T n(t)\, z(t)\, dt$$

$$= TC_{xz}(m) + N(T)$$

We allow m to be any one of a discrete number of values, and we assume the noise is white, with zero mean. The noise term at the termination of integration has variance

$$\sigma_N^2 = \mathcal{E}(N^2) = \int_0^T \int_0^T (N_0)/(2)\, \delta(t - s) \times z(t)\, z(s)\, dt ds$$

$$= (N_0)/(2) \int_0^T z^2(t)\, dt = \tfrac{1}{2} N_0 T$$

Let ΔC_{xz} represent the *distinguishability* of the normalized cross-correlation values $C_{xz}(m)$:

$$\Delta C_{xz} = |C_{xz}(m') - C_{xz}(m'')|$$

where $|C_{xz}(m')| \geq |C_{xz}(m)|$ for all m, and m'' is chosen to minimize the difference above. The distinguishability-to-noise ratio limits the error probability; that is, two correlation detectors will have approximately the same probability of error if they have the same distinguishability-to-noise ratio, $\mathcal{E}(\Delta\Lambda)/\sigma_N$:

$$[\mathcal{E}(\Delta\Lambda)]/(\sigma_N) = [T\Delta C_{xz}]/[(\tfrac{1}{2}N_0 T)^{\frac{1}{2}}]$$

$$= [(2T)/(N_0)]^{\frac{1}{2}} \Delta C_{xz}$$

As a result, the integration time for a given probability of error [more precisely, for a given $\mathcal{E}(\Delta\Lambda)/(\sigma_N)$] increases as the inverse-square of distinguishability of cross-correlation values.

$$T = \frac{N_0}{2} \frac{\mathcal{E}(\Delta\Lambda)}{\sigma_N} \left[\Delta C_{xz}\right]^{-2}$$

The ratio of the times T' and T for two such systems is hence

$$\frac{T'}{T} = \left(\frac{\Delta C_{xz}}{\Delta C_{x'z'}}\right)^2$$

C. Minimum Acquisition-Time Receivers

To minimize the acquisition ratio

$$\frac{T'_{acq}}{T_{acq}} = \frac{(v_1 + v_2 + \cdots + v_n)T'}{pT} = \frac{(v_1 + v_2 + \cdots + v_n)}{[v_1, v_2, \cdots, v_n]} \left(\frac{\Delta C_{xz}}{\Delta C_{x'z'}}\right)^2$$

for a fixed n, by choosing α and $\gamma_i, \cdots, \gamma_n$ properly, we must first make the distinguishabilities $\Delta C_{x'z'}$ as large as possible, and second, minimize $(v_1 + \cdots + v_n)/[v_1, \cdots, v_n]$.

Recall that for each i and j, v_i and v_j must have some non-unity relative prime factors. There will always exist v'_i, $i = 1, \cdots, n$, relatively prime in pairs (assuming $p \neq v_1 v_2 \cdots v_n$) with

$$p = v'_1 \, v'_2 \cdots v'_n$$

such that $(v'_1 + v'_2 + \cdots + v'_n) \leq (v_1 + v_2 + \cdots + v_n)$. To demonstrate that this is possible, we proceed as follows: stepwise, consider all pairs v_i, v_j, and arbitrarily set $v_i = v'_i$ and $v'_j = v_j/(v_i, v_j)$ at each step. The final set $\{v'_i\}$ is pairwise relatively prime and $v'_1 \, v'_2 \cdots v'_n = p$, with either $v'_i < v_i$ or $v'_i = v_i$. Hence, $(v_1 + v_2 + \cdots + v_n) \geq (v'_1 + v'_2 + \cdots v'_n)$.

Since we wish to pick v_i to minimize the acquisition ratio, we must let the v_i be relatively prime, for otherwise we could follow the procedure above to pick a relatively prime set of v'_i giving a smaller acquisition ratio.

It is a well-known result that $(v_1 + \cdots + v_n)$ is minimized, relative to the constraint that $p = v_1 \, v_2 \cdots v_n$, by choosing each v_i equal to $\sqrt[n]{p}$. Of course, the distinctness of each v_i makes this impossible. We must, in consolation, group the v_i as close to $\sqrt[n]{p}$ as possible, keeping them relatively prime.

In summary, for a minimum acquisition-time receiver, we seek n well-chosen sequences whose correlations $C_{\alpha\gamma_i}(m)$ have periods v_i which are relatively prime and close to $\sqrt[n]{p}$ and which have a maximum distinguishability $\Delta C'$ between phases. Over all such schemes, we then choose n to further minimize the acquisition ratio, approximately

$$\frac{T'_{acq}}{T_{acq}} \simeq n \, p^{\frac{1-n}{n}} \left(\frac{\Delta C}{\Delta C'}\right)^2$$

III. BOOLEAN COMBINATION OF COMPONENT SEQUENCES

It has been shown elsewhere (Ref. 7) that the distinguishability of an autocorrelation function can always be made greater than that of a cross-correlation function. We may attempt to minimize this effect by defining α as a combination of "component" sequences ξ_i. That is, we would like to be able to combine the ξ_i in some way to produce α, choosing this function to maximize the distinguishability. We are dealing with binary sequences, and it is thus natural to use a Boolean function. We will assume that, for an arbitrary Boolean function f, the function is applied termwise, as though α were the output of a switching network when the inputs are the ξ_i (see Fig. 3).

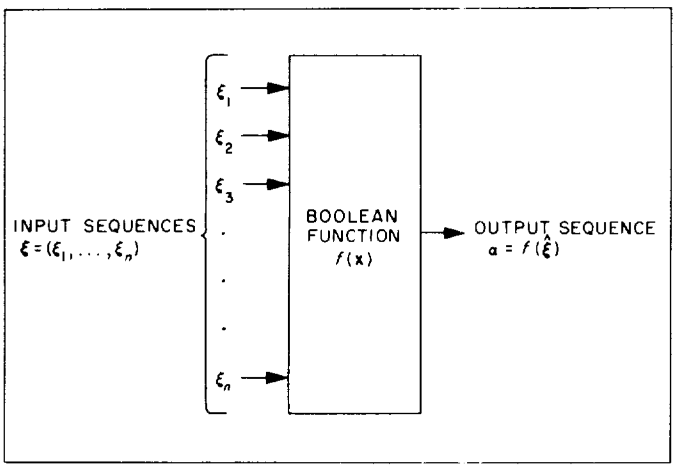

Fig. 3. Sequence generation by logical combination of component sequences

We assume α and the ξ_i are binary (± 1) sequences, so that f is a (± 1) Boolean function. Define $\hat{\alpha}$, $\hat{\xi}_i$, and $\hat{f}$ on $(0, 1)$:

$$\alpha_i = (-1)^{\hat{\alpha}_i}$$

$$\xi_{ij} = (-1)^{\hat{\xi}_{ij}}$$

$$f = (-1)^{\hat{f}}$$

As a convention, we will assume f is a (± 1) function of $(0, 1)$ variables x_i.

Since the transmitted sequence $\alpha = f(\hat{\boldsymbol{\xi}})$ is a function of component sequences, we will correlate α at the receiver against sequences $\gamma_i = g_i(\hat{\boldsymbol{\xi}})$, also made by forming Boolean functions of stored replicas of the same component sequences. When correlating α against γ_i, we agree to vary only the phase m_i of ξ_i at the receiver to compute $C_{\alpha\gamma_i}(m)$. Then $C_{\alpha\gamma_i}(m)$ has period $v_i = (p, u'_i)$, where u'_i is the period of ξ_i. By our reasoning in Section II-A, we see that u'_i must equal v_i and satisfy all the conditions laid forth previously.

To optimize the set of ξ_i, we thus must choose the v_i relatively prime in pairs, and each approximately $\sqrt[n]{p}$, where $p = v_1 v_2 \cdots v_n$ is the period of α. Then we must choose f and the g_i to make the $\Delta C_{\alpha\gamma_i}$ as large as possible.

A. The Boolean Transform

Let $f(\mathbf{x})$ be a ± 1-valued Boolean function of $(0, 1)$ variables $x_1, \cdots, x_n$. For any $\mathbf{s} = (s_1, s_2, \cdots, s_n)$, $s_i = 0$ or 1, define

$$\phi(\mathbf{s}, \mathbf{x}) = 2^{-n/2}(-1)^{s_1x_1 + \cdots + s_nx_n}$$

These 2^n functions of $\mathbf{x}$, the Rademacher-Walsh functions, form an orthonormal basis for 2^n-space. Since $f(\mathbf{x})$ is completely specified by the values it assumes on each of the 2^n different $\mathbf{x}$, f can be treated as a member of 2^n-space. Relative to the basis $\phi(\mathbf{s}, \mathbf{x})$, $f(\mathbf{x})$ has components $F(\mathbf{s})$ given by

$$F(\mathbf{s}) = 2^{-n/2} \sum_{\text{all } \mathbf{x}} f(\mathbf{x})\, \phi(\mathbf{s}, \mathbf{x})$$

That is, $F(\mathbf{s})$ is the projection of $f(\mathbf{x})$ on $\phi(\mathbf{s}, \mathbf{x})$, normalized so that

$$\sum_{\mathbf{s}} F^2(\mathbf{s}) = 1$$

Similarly,

$$f(\mathbf{x}) = 2^{n/2} \sum_{\text{all } \mathbf{s}} F(\mathbf{s})\, \phi(\mathbf{s}, \mathbf{x})$$

$F(\mathbf{s})$ can also be viewed as the correlation between the *truth-table* of $\hat{f}(\mathbf{x})$ and that of $s_1x_1 \oplus \cdots \oplus s_nx_n$.

B. The Correlation Function

Consider the effect of putting binary sequences ξ_i into the logic $f(\mathbf{x})$. The value of the ith sequence at time k is ξ_{ik}, and the vector giving the input variables to f at time k is $\boldsymbol{\xi}_k$; the output is then $f(\hat{\boldsymbol{\xi}}_k)$. This can also be expressed, by using the Kronecker delta, as

$$\alpha_k = f(\hat{\boldsymbol{\xi}}_k) = \sum_{\text{all } \mathbf{x}} f(\mathbf{x})\, \delta(\mathbf{x}, \hat{\boldsymbol{\xi}}_k)$$

$$\alpha_k = 2^{n/2} \sum_{\text{all } \mathbf{s}} F(\mathbf{s})\, \phi(\mathbf{s}, \hat{\boldsymbol{\xi}}_k)$$
$$= \sum_{\text{all } \mathbf{s}} F(\mathbf{s}) (-1)^{s_1 \hat{\xi}_{1k} + \cdots + s_n \hat{\xi}_{nk}}$$
$$= \sum_{\text{all } \mathbf{s}} F(\mathbf{s})\, \xi_{1k}^{s_1} \xi_{2k}^{s_2} \cdots \xi_{nk}^{s_n}$$

Note in the above that the $F(\mathbf{s})$ are properties of the logic alone and do not involve the form of the input sequences.

Let γ be another sequence made by inserting the n component sequences ξ_i into a logic $g(\mathbf{x})$:

$$\gamma_k = g(\hat{\boldsymbol{\xi}}_k) = \sum_{\mathbf{s}} G(\mathbf{s})\, \xi_{1k}^{s_1} \xi_{2k}^{s_2} \cdots \xi_{nk}^{s_n}$$

At the receiver, we correlate the incoming sequence α against this locally generated one, choosing the logic $g(\mathbf{x})$ to maximize the distinguishability among phases. We allow different logics at the receiver for each component to be acquired; that is, while ξ_1 is being acquired, use $g_1(\mathbf{x})$, and while ξ_2 is being acquired, use $g_2(\mathbf{x})$, etc. This cross-correlation takes the form

$$C_{\alpha\gamma}(m) = \sum_{\mathbf{s}} \sum_{\mathbf{w}} F(\mathbf{s})\, G(\mathbf{w}) \left[\frac{1}{p} \sum_{k=0}^{p-1} \xi_{1,k}^{s_1} \xi_{1,k+m}^{w_1} \cdots \xi_{n,k}^{s_n} \xi_{n,k+m}^{w_n} \right]$$

The fact that the ξ_i all have relative prime periods means that they are independent to the degree that the correlation of products is the product of the correlations. Because of this product rule and because

$$\frac{1}{v_i} \sum_{k=0}^{v_i-1} \xi_{i,k}^{s_i} \xi_{i,k+m}^{w_i} = \begin{cases} 1 & \text{if } s_i = w_i = 0 \\ d_i & \text{if } s_i \neq w_i \\ C_i(m) & \text{if } s_i = w_i = 1 \end{cases}$$

where we introduce the notation

$$d_i = \frac{1}{v_i} \sum_{k=0}^{v-1} \xi_{ik}$$

$$C_i(m) = \frac{1}{v_i} \sum_{k=0}^{v-1} \xi_{i,k}\, \xi_{i,k+m}$$

we can then write the following expression for the cross-correlation function

$$C_{\alpha\gamma}(m) = \sum_{\mathbf{s}} \sum_{\mathbf{w}} F(\mathbf{s})\, G(\mathbf{w}) \prod_{i=1}^{n} d_i^{|s_i - w_i|} [C_i(m)]^{s_i w_i}$$

(Here we adopt $[\cdot]^0 = 1$ purely as convention.) If we denote the product term as $C(m; \mathbf{s}, \mathbf{w})$,

$$C_{\alpha\gamma}(m) = \sum_{\mathbf{s}} \sum_{\mathbf{w}} F(\mathbf{s})\, G(\mathbf{w})\, C(m; \mathbf{s}, \mathbf{w})$$

This formula is of fundamental importance in finding the minimum acquisition-time receiver. Note that, by using it, one may express the cross-correlation between any two Boolean functions of the ξ_i as a sum of transform coefficients of the two functions weighted by autocorrelation properties of the ξ_i. Also note in the equation $C(m; \mathbf{s}, \mathbf{w})$ that when $s_j \neq w_j$,

$$|C(m; \mathbf{s}, \mathbf{w})| \leq d_j$$

and when both $s_j \neq w_j$ and $s_i \neq w_i$,

$$|C(m; \mathbf{s}, \mathbf{w})| \leq d_i d_j, \text{ etc.}$$

From these considerations, when the d_j are sufficiently small, we may often omit the terms with $\mathbf{s} \neq \mathbf{w}$ from the correlation equation. This is generally the case, for as we shall see, the ξ_i must have maximally distinguishable correlation functions, a condition requiring small d_i (Ref. 7).

IV. MINIMUM ACQUISITION-TIME SYSTEMS

In this Section we treat two ranging systems. The first is the optimal configuration assuming that symbol synchronization between transmitter and receiver is achieved through some external source. The second relaxes this condition, but instead uses one of the components as a "clock" sequence, locking a phase-locked loop to the incoming symbol rate.

Not only are the encoding logics f different for these types of systems, but also the decoding logics g_i.

We assume that the transmitter codes are displaced from the receiver codes by $m_1, m_2, \cdots, m_n$ steps, or an equivalent shift of m steps,

$$m = m_i \; (\mathrm{mod}\; v_i)$$

$$v_i = \text{period of sequence at input } x_i$$

We step each component at the receiver until we have moved the locally generated components by an amount equivalent to m and thereby determine the "distance" m from transmitter to receiver.

If the α received is delayed by m steps, our decoding scheme is also clear: after having found the delays m_i giving maximum cross-correlations of α with each of ξ_i, we declare that m is that integer such that, for each i,

$$m = m_i \; (\mathrm{mod}\; v_i)$$

which has a unique solution modulo p by the "Chinese" remainder theorem (Ref. 8) of number theory.

A. The Synchronous Receiver

This first system, as we have indicated above, is based on the assumption that initial synchronization or clock-lock is in effect so that the received sequence is symbol-wise in-step with the locally generated ones. The decoding procedure is based on the following criteria:

1. The order of component acquisition is immaterial, provided the proper decoding logic corresponding to that component is used.
2. Acquisition of any component does not rely on prior acquisition of any other component.

Consider the terms in $f(\hat{\boldsymbol{\xi}})$ involving ξ_i; i.e., the contribution of the ith input to the total output. Call this part of the signal $f_i(\hat{\boldsymbol{\xi}})$. For example,

$$f_1(\hat{\boldsymbol{\xi}}) = F(1, 0, \cdots, 0)\, \xi_1 + F(1, 1, 0, \cdots 0)\, \xi_1 \xi_2 + \cdots$$

Those terms of f_1 in which $\xi_2, \cdots, \xi_n$ appear can be viewed as "cross-talk," which can be a degrading factor in determining the shift of ξ_1 if no knowledge of $\xi_2, \cdots, \xi_n$ is assumed.

The receiver must thus either minimize this cross-talk and/or try to estimate what the cross-talk will be and use this information to further enhance reception.

If the signals in each channel are independent, or if we do not allow estimation of one component to influence the estimation of another, we have no other course than to minimize cross-talk. We desire, then, to separate from $f(\hat{\boldsymbol{\xi}})$ only that component carrying the information we want. This is accomplished most effectively by correlating $\alpha = f(\hat{\boldsymbol{\xi}})$ against ξ_1. We desire to pick f and the ξ_i, $i = 1, 2, \cdots, n$, in such a way that the cross-correlations of α with each ξ_i have maximum distinguishability. By choosing $g_i(\mathbf{x}) = (-1)^{x_i}$, we can write ξ_i as

$$\xi_i = g_i(\hat{\boldsymbol{\xi}})$$

The transform of g_i is easily computed, for we note that $g_i(\mathbf{x}) = 2^{n/2}\phi(\mathbf{x}, \mathbf{e}^i)$, defining $\mathbf{e}^i$ to be the ith unit vector having a single one, in the ith place.

$$\mathbf{e}^i = (0, 0, \cdots, 0, 1, 0, \cdots, 0)$$

The transform of g_i is then

$$G_i(\mathbf{s}) = \delta\,(\mathbf{s}, \mathbf{e}^i)$$

Consequently, the cross-correlation equation reduces to

$$C_{\alpha\xi_i}(m) = \left\{ \left[\sum_{\mathbf{s}, s_i = 1} F(\mathbf{s}) \prod_{j \neq i} (d_j)^{s_j} \right] C_i\,(m) + \left[\sum_{\mathbf{s}, s_i = 0} F(\mathbf{s}) \prod_{j \neq i} (d_j)^{s_j} \right] d_i \right\}$$

For any two values m' and m'' of m_i, the difference in correlation values $C_{\alpha\xi_i}\,(m)$ (and specifically the distinguishability) is dependent separately on the autocorrelation of ξ_i and the Boolean function

$$C_{\alpha\xi_i}(m') - C_{\alpha\xi_i}(m'') = \left[\sum_{\mathbf{s}, s_i = 1} F(\mathbf{s}) \prod_{j \neq i} (d_j)^{s_j} \right] \times \left[C_i(m') - C_i(m'') \right]$$

Our course to optimize the acquisition receiver is now clear; first, each ξ_i is to have minimum out-of-phase autocorrelation values so that $C_i(m)$ has maximum distinguishability and, second, f is to be chosen such that

$$\left| \sum_{\mathbf{s}, s_i = 1} F(\mathbf{s}) \prod_{j \neq i} (d_j)^{s_j} \right|$$

is maximized—also for each i. Further, we can always choose the sum to be *positive* by proper choice of f; for suppose the sum were negative. By choosing $g'_i(\mathbf{s}) = (-1)^{x_i+1}$, we correlate α against ξ'_i, given by

$$\xi'_i = g'_i(\hat{\boldsymbol{\xi}}) = -\xi_i$$

and have $C_{\alpha\xi'_i}(m) = -C_{\alpha\xi_i}(m)$, which has the sum in question positive. By duality, we can thus always complement x_i in $f(\mathbf{x})$, if need be, to make

$$\sum_{\mathbf{s}, s_i = 1} F(\mathbf{s}) \prod_{j \neq i} (d_j)^{s_j} \geq 0$$

Now, consider the sum to be maximized,

$$\sum_{\mathbf{s}, s_i = 1} F(\mathbf{s}) \prod_{j \neq i} (d_j)^{s_j}$$

One of the terms in the sum is $F(\mathbf{e}^i)$, but the remainder have products of d_j as factors. Denote

$$d' = \max_{i \neq j} \{ |d_j| \}$$

$$F_M = \max_{\substack{\mathbf{s}, s_i = 1 \\ \mathbf{s} \neq \mathbf{e}^i}} \{ |F(s)| \}$$

Using the triangle inequality, we bound the sum of remaining terms, calling it F, as follows:

$$F = \left| \sum_{\substack{\mathbf{s}, s_i = 1 \\ \mathbf{s} \neq \mathbf{e}^i}} F(\mathbf{s}) \prod_{j \neq i} (d_j)^{s_j} \right| \leq \sum_{\substack{\mathbf{s}, s_i = 1 \\ \mathbf{s} \neq \mathbf{e}^i}} |F(\mathbf{s})| \prod_{j \neq i} |d_j|^{s_j}$$

$$|F| \leq F_M \sum_{\substack{\mathbf{s}, s_i = 1 \\ \mathbf{s} \neq \mathbf{e}^i}} \prod_{j \neq i} (d')^{s_j}$$

$$\leq F_M \left[\sum_{k=0}^{n-1} \binom{n-1}{k} (d')^k - 1 \right]$$

The binominal theorem applied to the inequality gives

$$\left| \sum_{\mathbf{s}, s_i = 1} F(\mathbf{s}) \prod_{j \neq i} (d') \right| \leq F_M \left[(1 + d')^{n-1} - 1 \right]$$

Note that when d' is small this upper bound can be replaced by $nd'F_M$:

$$|F| \leq nd'F_M$$

We recognize that by using nearly balanced sequences for the ξ_i [which we want to do to optimize distinguishability (Ref. 7)], it is highly efficient to maximize $F(\mathbf{e}^i)$ by proper choice of $f(\mathbf{x})$. In fact, any time that $nd' < 1$, this is the course we *must* follow to insure the largest possible $F(\mathbf{e}^i) + F$. When we use nearly balanced sequences, we can approximate

$$C_{\alpha\xi_i}(m) \simeq F(\mathbf{e}^i)\, C_i(m)$$

So that all channels are identical, let us set all $F(\mathbf{e}^i) = F(\mathbf{e}^1)$. We can then prove that *if* $f(\mathbf{x})$ *is a Boolean function such that* $F(\mathbf{e}^i) = F(\mathbf{e}^1)$ *for all* i, *and* $F(\mathbf{e}^1)$ *is maximal over all Boolean functions, then* $f(\mathbf{x})$ *is a strict majority logic*:

$$f(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \text{ has less than } n/2 \text{ one's} \\ -1 & \text{if } \mathbf{x} \text{ has more than } n/2 \text{ one's} \end{cases}$$

To show that this is true, note that

$$F(\mathbf{e}^1) = \frac{1}{n} \sum_{i=1}^{n} F(\mathbf{e}^i) = \frac{2^{-n}}{n} \sum_{i=1}^{n} \sum_{\mathbf{x}} f(\mathbf{x}) (-1)^{x_i}$$

$$= \frac{2^{-n}}{n} \sum_{\mathbf{x}} f(\mathbf{x}) \sum_{i=1}^{n} (-1)^{x_i}$$

$$= \frac{2^{-n}}{n} \sum_{\mathbf{x}} f(\mathbf{x}) \Big[n - 2 \|\mathbf{x}\| \Big]$$

where $\|\mathbf{x}\|$ denotes the number of *one's* in $\mathbf{x}$. To maximize $F(\mathbf{e}^1)$, if $\|\mathbf{x}\| > n/2$, we must make $f(\mathbf{x}) = -1$, and if $\|\mathbf{x}\| < n/2$, we must make $f(\mathbf{x}) = 1$. Those $\mathbf{x}$ with $\|\mathbf{x}\| = n/2$, if n is even, may be placed arbitrarily in the truth-table of f without affecting $F(\mathbf{e}^i)$. If n is odd, f is a symmetric Boolean function; that is, we may permute the x_i without changing f. And if n is even, we can make it symmetric by symmetric placement of those $\mathbf{x}$ with $\|\mathbf{x}\| = n/2$ in the truth-table. Then, if $[y]$ denotes the integer part of the number y,

$$F(\mathbf{e}^i) = \frac{2^{1-n}}{n} \sum_{k=1}^{[n/2]} \binom{n}{k} [n - 2k]$$

$$= 2^{1-n} \binom{n-1}{\left[\frac{n}{2}\right]}$$

For moderately large n, this is approximately

$$F(\mathbf{e}^i) \simeq \Big[(\pi/2)(n-1) \Big]^{-\frac{1}{2}}$$

by the Stirling formula (Ref. 2).

Into the acquisition ratio we insert the distinguishability for the cross-correlations $C_{\alpha\xi_i}(m)$ and the distinguishability of $C_{\alpha'}(m)$ if α' were an optimal sequence from the autocorrelation viewpoint. Whenever the v_i are much larger than unity, both ΔC_i and ΔC_a are approximately one. The acquisition ratio to be minimized is, then, approximately given by

$$\frac{T'_{\mathrm{acq}}(n)}{T_{\mathrm{acq}}} \simeq n\, p^{\frac{1-n}{n}} \left[2^{1-n} \binom{n-1}{\frac{n-1}{2}} \right]^{-2}$$

(As shown in Appendix B, we need consider only *odd n.*) Use of the Stirling approximation reduces the approximate acquisition ratio to

$$\frac{T'_{\mathrm{acq}}}{T_{\mathrm{acq}}} \simeq \left(\frac{\pi}{2p}\right) n(n-1)\, p^{1/n}$$

To find the optimum value of n, the derivative of $T'_{\mathrm{acq}}/T_{\mathrm{acq}}$,

$$\frac{d(T'_{\mathrm{acq}}/T_{\mathrm{acq}})}{dn} \simeq \left(\frac{\pi}{2p}\right)\left(\frac{p^{1/n}}{n}\right)[2n^2 - n - (n-1)\ln p]$$

goes to zero only when the term in brackets is zero; this occurs at those values of n, such that

$$\ln p = [(2n-1)n]/[n-1]$$

$$p = e^{\frac{n(2n-1)}{n-1}}$$

$$v_i = \sqrt[n]{p} = e^2\, e^{\frac{1}{n+1}}$$

Upon insertion of this value into the acquisition ratio, we find the optimal ratio:

$$\left(\frac{T'_{\mathrm{acq}}}{T_{\mathrm{acq}}}\right)_{\mathrm{opt}} \simeq \frac{\pi}{2}\, n(n-1)\, e^{-2n+1}$$

This ratio above is tabulated in Table 1. Note that the ratio is less than unity, and hence the minimal acquisition-time receiver is better than matched filters.

Hence, the minimal acquisition-time receiver would, ideally, given an α-period p, combine n optimal binary sequences with

$$p \simeq e^{\frac{n(2n-1)}{n-1}}$$

using component sequences ξ_i of periods v_i relatively prime in pairs and near to 9($\simeq e^2$).

Table 1. Optimal acquisition ratio and periods for given *n*, single correlator case

n	p	$\left(\frac{T'_{\mathrm{acq}}}{T_{\mathrm{acq}}}\right)$
1	any	1.0×10^{0}
3	1.8×10^{3}	6.4×10^{-2}
5	7.6×10^{4}	3.9×10^{-3}
7	3.8×10^{6}	1.5×10^{-5}
9	2.0×10^{8}	4.7×10^{-6}
11	1.0×10^{10}	1.3×10^{-8}
13	5.7×10^{11}	3.4×10^{-9}
15	3.1×10^{13}	8.4×10^{-11}
17	1.6×10^{15}	2.0×10^{-12}
19	9.1×10^{16}	4.6×10^{-14}

B. Modified Synchronous Receivers

Suppose, as an alternative, we are willing to make a receiver which has one correlator for each of the components ξ_i of α. What is the best receiver? Just as in the constant-equipment case, we define an acquisition ratio:

$$\frac{T'_{\mathrm{acq}}}{T_{\mathrm{acq}}} = \frac{\text{time for } n\text{-component acquisition}}{\text{time for 1-component acquisition}}$$

The time for a 1-component code α to be acquired is merely its period p times the integration time T per phase, or pT. On the other hand, with n correlators working simultaneously, the time to be acquired is the new integration time per step T' times the number of phases, or $\max_i \{v_i\}\, T'$,

$$\frac{T'_{\mathrm{acq}}}{T_{\mathrm{acq}}} = \frac{\max\{v_i\}}{[v_1, v_2, \cdots, v_n]}\,\frac{T'}{T}$$

To minimize this ratio, we may argue as before: the v_i must be relatively prime; for if they were not, we could pick a relatively prime set with the same least common multiple but having a smaller maximum component. Next, to further minimize the ratio, we want to make $(v_i)_{\max}$ as close to the average v_i as possible

$$(v_i)_{\max} \simeq (v_1 + \cdots + v_n)/(n)$$

The best acquisition ratio is thus given by

$$\frac{T'_{\rm acq}}{T_{\rm acq}} \simeq \frac{v_1 + \cdots + v_n}{n\, v_1\, v_2 \cdots\, v_n} \frac{T'}{T}$$

This equation is exactly the same form as that for the minimum-equipment receiver described previously except for a factor of $1/n$. The same technique for obtaining α from the components ξ_i (which must be optimum binary sequences) must be applied in both cases; that is, $\hat{\alpha} = \text{maj}\,(\hat{\xi})$. Further,

$$v_i \simeq \sqrt[n]{p}$$

With a majority logic, optimum components, and $v_i \simeq \sqrt[n]{p}$, the acquisition ratio is approximately $1/n$ times that found in Section IV-A:

$$\frac{T'_{\rm acq}}{T_{\rm acq}} \simeq p^{-1+1/n} \left[2^{-n+1} \begin{pmatrix} n-1 \\ \frac{n-1}{2} \end{pmatrix} \right]^{-2}$$

Upon setting the derivative of this ratio to zero, we find

$$p \simeq e^{\frac{n^2}{n-1}}$$

$$v_i \simeq e^{\frac{n}{n-1}} \simeq e$$

$$\left(\frac{T'_{\rm acq}}{T_{\rm acq}}\right)_{\rm opt} \simeq \frac{\pi}{2}\,(n-1)\,e^n$$

This is tabulated in Table 2. Although if each v_i were about $3(\simeq e)$ in length, the analysis above, based upon the assumptions that the d_i are small and n is large, may not be strictly valid because the relative prime condition on $\{v_i\}$ may carry $(v_i)_{\max}$ far from e. But the analysis is indicative of the action to be taken in the design of such a receiver; after an approximate choice of p, choose n such that

$$p \simeq e^{\frac{n^2}{n-1}}$$

Having this n, choose n relatively prime optimal components ξ_i whose periods are as *small* (but greater than one) as possible. Then modify the choice of p to

$$p = v_1\, v_2 \cdots\, v_n$$

The approximations certainly establish a lower bound on the acquisition ratio, in any case, since optimal conditions were assumed at all times.

Table 2. Optimal acquisition ratio and periods for given n, n-correlator case

n	p	$\left(\frac{T'_{\rm acq}}{T_{\rm acq}}\right)$
1	any	1.0×10^{0}
3	9.0×10	1.5×10^{-1}
5	5.2×10^{2}	4.2×10^{-2}
7	3.5×10^{3}	8.6×10^{-3}
9	2.5×10^{4}	1.6×10^{-3}
11	1.8×10^{5}	2.6×10^{-4}
13	1.3×10^{6}	4.3×10^{-5}
15	9.5×10^{6}	6.7×10^{-6}
17	7.0×10^{7}	1.0×10^{-6}
19	5.1×10^{8}	1.6×10^{-7}

C. Optimal Clock-Component Codes

We have previously assumed an initial synchronization or clock-lock condition in finding optimal codes. However, we now relax this condition so that the receiver must not only determine the proper integral number of phase steps separating the incoming and local codes, but also it must acquire the incoming symbol rate and lock its code generators to it.

Easterling's single-channel ranging receiver is shown in Fig. 4. The inner loop, or clock-loop, is synchronized to the symbol rate of the incoming code α by the presence of a "clock component" in α, and the locally generated code γ is slaved to the output of this clock-loop.

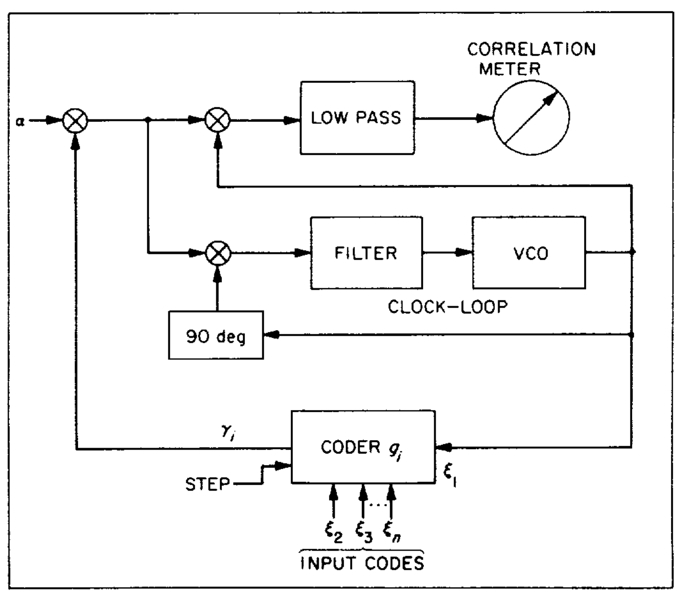

Fig. 4. The single channel ranging receiver with clock-acquiring loop

for every binary vector $(s_2, s_3, \cdots, s_n)$. Now choose the u vectors on which $\hat{g}_k(x_1, \cdots, x_n) = 1$ as follows:

(1) $x_k = 1$ for each of the u vectors

(2) if for any $(s_2, \cdots, s_n)$ having all $s_j = 0, j > k$, and $F'(\mathbf{s}) \neq 0$, make the number

$$G'_k(s_2, \cdots, s_n) = 2^{1-n} \left\{ \begin{array}{l} \text{difference between number} \\ \quad \text{of times} \\ s_2x_2 \oplus \cdots \oplus s_nx_n = 1 \text{ and} \\ s_2x_2 \oplus \cdots \oplus s_nx_n = 0 \end{array} \right\}$$

have the same sign as $F'(\mathbf{s})$ and be as large in magnitude as possible; if it is not possible to make them have the same sign, make $G'_k(\mathbf{s})$ as near to zero as possible.

(3) If any $\mathbf{s}$ has $s_j = 1, j > k$, and $F'(\mathbf{s}) \neq 0$, make $G'_k(\mathbf{s})$ equal to zero if possible; and if not, preferably $F'(\mathbf{s})$ and $G'_k(\mathbf{s})$ should have the same sign.

The reasons for the above steps are based on the fact that, according to the assumptions, it follows that

$$\Delta C_{a\gamma_k}(m) = C_0K_0 + \sum_{\mathbf{s}\neq 0} F'(\mathbf{s})\, G'_k(\mathbf{s}) \prod_{i=2}^{n} [C_i(m)]^{s_i}$$

Step (1) makes the principal jump as large as possible, step (2) asks that the previously acquired components enhance acquisition when possible, and step (3) insures that the effect of unacquired components is minimized.

APPENDIX A

Example of an Optimal Clock-Component System

Consider a five-component (four-component plus clock) system with the following constraints:

1. Initial clock-lock, $C_0 = 0.625$
2. Drop in clock-lock, $K_0 = 0.625$

These numbers fix the number of one's in the truth-tables of $\hat{f}$ and $\hat{g}_i$:

$$w = u = 2^4(1 - 0.625) = 6$$

The six one's of $\hat{f}$ are then placed on the following vectors, in accordance with the modified majority logic:

$$\hat{f}: \left\{ \begin{array}{l} (1\,1\,1\,1\,1) \\ (1\,1\,1\,1\,0) \\ (1\,1\,1\,0\,1) \\ (1\,1\,0\,1\,1) \\ (1\,0\,1\,1\,1) \\ (0\,1\,1\,1\,1) \end{array} \right.$$

Whenever the clock-loop is locked to the clock component of α, the local code γ is stepwise synchronized to α.

Both α and γ are logical combinations of a clock sequence

$$+ - + - + - + - + - \cdots$$

and $n - 1$ other sequences, whose properties we shall describe in more detail later. The transmitter logic we take, for convenience, to be of the form

$$x_1 \oplus \hat{f}(x_1, x_2, \cdots, x_n)$$

and, similarly, those at the receiver to be

$$x_1 \oplus \hat{g}_i(x_1, x_2, \cdots, x_n)$$

In these functions, x_1 is the clock input and $x_2, \cdots, x_n$ are the other sequence inputs; $\oplus$, as usual, indicates modulo 2 addition. We have indexed the g's with an i to denote that we are willing to use different logics in the decoding procedure, perhaps a different logic for each component.

The correlation meter reads the normalized cross-correlation $C(m)$ between the incoming and local codes; the clock-loop is held in lock according to the normalized slope of $C(m)$, defined as

$$\Delta C(m) = \tfrac{1}{2}\,[C(m) - C(m')]$$

where m' is the effect of stepping the clock phase one step forward; i.e., from m_1 to $m_1 + 1$. We shall refer to $\Delta C(m)$ as the *clock-lock correlation*.

The optimal codes to use in the above scheme must have the following properties: (1) A specified initial clock-lock correlation, (2) maximum increase in clock-lock correlation as components are acquired, and (3) no more than a specified percentage drop in clock-lock during the search.

Besides these obvious requirements, it is advantageous to adopt the following philosophy in choosing the logics:

1. Acquisition of kth component does not *rely* on the fact that any of the $k - 2$ previously considered non-clock components have been acquired.
2. The fact that previously considered components *are* acquired shall enhance acquisition of the component under present scrutiny.
3. Components not yet considered, whether such components happen to be already in-phase or not, shall not affect the acquisition.

In previous analysis, we have seen that the components ξ_i must have relative prime periods, have small out-of-phase correlation, and a balance of one's and minus one's. To simplify calculations, we make the following assumptions concerning the ξ_i:

1. Independence; i.e., the correlation between the ith and the jth input sequence $|C_{ij}(m)| = 0$ for all m, every $i \neq j$ and $i, j \neq 1$.
2. Perfect autocorrelation of input sequences; i.e., $C_i(m) = 0$ for all $m \not\equiv 0 \bmod v_i$, all $i \neq 1$.
3. Balance; i.e., equal number of *one's* and *zero's* in each component sequence, per period.

For the type of sequences we must use as components (pseudonoise or near-pseudonoise), none of the above assumptions strictly applies—in fact, 1 and 2 cannot occur simultaneously. However, each "$= 0$" above can be replaced by "$< \epsilon$" for some appropriate ϵ, so that the results are essentially the same whenever ϵ is small.

Based on these criteria, the optimum coding and decoding functions are found by the following rules:

1. *Encoding function:* transmit $x_1 \oplus \hat{f}(x_1, x_2, \cdots, x_n)$, where f has w one's in its truth table,

$$w = 2^{n-1}(1 - C_0)$$

$$C_0 = \text{initial clock-lock correlation}$$

These w one's are put in f in a modified majority logic:

if $\hat{f}(x_1, x_2, \cdots, x_n) = 1$

and $(x_2, \cdots, x_n)$ has fewer one's than some $(y_2, \cdots, y_n)$

then $\hat{f}(y_1, y_2, \cdots, y_n) = 1$

2. *Decoding functions:* decode by correlating with $x_1 \oplus \hat{g}_i(x_1, x_2, \cdots, x_n)$ where each $\hat{g}_i$ has u one's in its truth-table,

$$u = 2^{n-1}(1 - K_0)$$

$$K_0 = \text{fractional drop in clock-lock from } C_0$$

a. *Clock-component acquisition:* $\hat{g}_i(x_1, x_2, \cdots, x_n) = 0$
$\Delta C = C_0$

b. *kth-component acquisition:* first list the w vectors on which $\hat{f}(x_1, x_2, \cdots, x_n) = 1$ and calculate the numbers

$$F'(s_2, \cdots, s_n) = 2^{1-n}\left\{\begin{array}{l}\text{difference between number} \\ \quad\text{of times} \\ s_2x_2 \oplus \cdots \oplus s_nx_n = 1 \text{ and} \\ s_2x_2 \oplus \cdots \oplus s_nx_n = 0\end{array}\right\}$$

The 16 numbers $F'(\mathbf{s})$ are easily calculated:

$$F'(1000) = F'(0100) = F'(0010) = F'(0001) = 0.25$$
$$F'(1100) = F'(1010) = F'(1001) = F'(0110) = F'(0101) = F'(0011) = -0.125$$
$$F'(1110) = F'(1101) = F'(1011) = F'(0111) = 0$$
$$F'(1111) = 0.125$$

To design an optimal clock-component system, we utilize the rules stated in Section IV-C:

1. *Clock acquisition:* $\Delta C_1 = C_0 = 0.625$

2. *Second component:* We choose six vectors which satisfy the rules (1), (2) and (3):

By (1): (- 1 - - -)
(- 1 - - -)
(- 1 - - -)
(- 1 - - -)
(- 1 - - -)
(- 1 - - -)

By (2): no constraint

By (3): columns 3, 4, and 5 must be balanced. Modulo 2 sum of columns 3, 4, and 5 should have balance or excess of one's.

Since there are six vectors, it is not possible to have rows 3, 4, and 5 balanced *and* the modulo 2 sum of any two columns also balanced. Hence, we seek to have more zero's in the modulo 2 sums of any two columns.

$$\hat{g}_2: \left\{\begin{array}{l}(01000)\\(11001)\\(01010)\\(01111)\\(11111)\\(11100)\end{array}\right.$$

$$G_2'(1000) = 0.375$$
$$G_2'(0100) = G_2'(0010) = G_2'(0001) = 0$$
$$G_2'(1100) = G_2'(1010) = G_2'(1001) = 0$$
$$G_2'(0110) = G_2'(0011) = G_2'(0101) = -0.125$$
$$G_2'(1110) = G_2'(1011) = G_2'(1101) = 0.125$$
$$G_2'(0111) = 0.25$$
$$G_2'(1111) = -0.25$$

The only $G_2'(\mathbf{s})$ not satisfying (3) is $G_2'(1111)$. But we reason that this is acceptable because its contribution is detrimental only when all components are already in lock, a likelihood of only one chance per total period.

The clock-lock correlation equation is

$$\Delta C_2(m) = 0.391 + 0.094\, C_2(m) + 0.015\, [C_3(m)\, C_4(m) + C_3(m)\, C_5(m) + C_4(m)\, C_5(m)] - 0.031\, C_2(m)\, C_3(m)\, C_4(m)\, C_5(m)$$

Note from this that even if all components were in-phase, the net result would be an enhancement of 1.5%.

3. *Third component:* Again choose six vectors:

By (1): (- - 1 - -)
(- - 1 - -)
(- - 1 - -)
(- - 1 - -)
(- - 1 - -)
(- - 1 - -)

By (2): G_3'(- - 0 0) is to have the same sign as F'(- - 0 0) and be as large as possible:

(- 1 1 - -)
(- 1 1 - -)
(- 1 1 - -)
(- 1 1 - -)
(- 1 1 - -)
(- 1 1 - -)

By (3): Again, columns 4 and 5 must be balanced. This again means their modulo 2 sum cannot be balanced. Since $F'(0011)$ is negative, we must have a majority of zero's in $x_4 \oplus x_5$:

$$\hat{g}_3: \left\{\begin{array}{l}(01100)\\(11100)\\(01101)\\(01111)\\(11111)\\(11110)\end{array}\right.$$

$$G'_3(1\,0\,0\,0) = G'_3(0\,1\,0\,0) = 0.375$$
$$G'_3(0\,0\,1\,0) = G'_3(0\,0\,0\,1) = 0$$
$$G'_3(1\,1\,0\,0) = -\,0.375$$
$$G'_3(1\,0\,1\,0) = G'_3(1\,0\,0\,1) = G'_3(0\,1\,1\,0) = G'_3(0\,1\,0\,1) = 0$$
$$G'_3(1\,1\,0\,1) = G'_3(1\,1\,1\,0) = 0$$
$$G'_3(0\,0\,1\,1) = G'_3(1\,1\,1\,1) = -\,0.125$$
$$G'_3(1\,0\,1\,1) = G'_3(0\,1\,1\,1) = 0.125$$

The clock-lock correlation equation is

$$\Delta C_3(m) = 0.391 + 0.094\,C_2(m) + 0.094\,C_3(m) + 0.047\,C_2(m)\,C_3(m) + 0.015\,C_4(m)\,C_5(m) - 0.015\,C_2(m)\,C_3(m)\,C_4(m)\,C_5(m)$$

Again, the effect of components 4 and 5 being in-phase is nullified.

4. *Fourth component:* The six vectors on which $\hat{g}_4 = 1$ must have:

By (1):
$$\begin{array}{c}(-\ -\ -\ 1\ -)\\(-\ -\ -\ 1\ -)\\(-\ -\ -\ 1\ -)\\(-\ -\ -\ 1\ -)\\(-\ -\ -\ 1\ -)\\(-\ -\ -\ 1\ -)\end{array}$$

By (2): $G'_4(-\ -\ -\ 0)$ are to have the same sign as $F'(-\ -\ -\ 0)$ and be as large as possible.

$$\begin{array}{c}(-\ 1\ 1\ 1\ -)\\(-\ 1\ 1\ 1\ -)\\(-\ 1\ 1\ 1\ -)\\(-\ 1\ 1\ 1\ -)\\(-\ 1\ 0\ 1\ -)\\(-\ 1\ 0\ 1\ -)\end{array}$$

By (3): Column 5 is to be balanced, and since $F'(0\,1\,0\,1)$ is non-zero, we make $G'_4(0\,1\,0\,1) = 0$:

$$\hat{g}_4: \left\{\begin{array}{c}(0\,1\,1\,1\,0)\\(0\,1\,1\,1\,1)\\(1\,1\,1\,1\,0)\\(1\,1\,1\,1\,1)\\(0\,1\,0\,1\,0)\\(1\,1\,0\,1\,1)\end{array}\right.$$

$$G'_4(1\,0\,0\,0) = G'_4(0\,0\,1\,0) = 0.375$$
$$G'_4(1\,0\,1\,0) = -\,0.375$$
$$G'_4(0\,1\,0\,0) = G'_4(1\,1\,1\,0) = 0.125$$
$$G'_4(1\,1\,0\,0) = G'_4(0\,1\,1\,0) = -\,0.125$$
$$G'_4(0\,0\,0\,1) = G'_4(1\,0\,0\,1) = G'_4(0\,0\,1\,1) = G'_4(1\,0\,1\,1) = 0$$
$$G'_4(0\,1\,0\,1) = G'_4(1\,1\,0\,1) = G'_4(0\,1\,1\,1) = G'_4(1\,1\,1\,1) = 0$$

Clock correlation is now

$$\Delta C_4(m) = 0.391 + 0.094\,C_2(m) + 0.031\,C_3(m) + 0.094\,C_4(m) + 0.015\,C_2(m)\,C_3(m) + 0.047\,C_2(m)\,C_4(m) + 0.015\,C_3(m)\,C_4(m)$$

5. *Fifth component:* The last six vectors must satisfy only (1) and (2):

By (1):
(- - - - 1)
(- - - - 1)
(- - - - 1)
(- - - - 1)
(- - - - 1)
(- - - - 1)

By (2):

$$\hat{g}_5: \begin{cases} (0\,1\,1\,1\,1) \\ (1\,1\,1\,1\,1) \\ (0\,1\,0\,1\,1) \\ (1\,1\,0\,1\,1) \\ (0\,1\,1\,0\,1) \\ (1\,1\,1\,0\,1) \end{cases}$$

$$G'_5(1\,0\,0\,0) = G'_5(0\,0\,0\,1) = 0.375$$
$$G'_5(0\,1\,0\,0) = G'_5(0\,0\,1\,0) = G'_5(1\,1\,0\,1) = G'_5(1\,0\,1\,1) = 0.125$$
$$G'_5(1\,1\,0\,0) = G'_5(1\,0\,1\,0) = G'_5(0\,1\,0\,1) = G'_5(0\,0\,1\,1) = -0.125$$
$$G'_5(1\,0\,0\,1) = -0.375$$
$$G'_5(0\,1\,1\,0) = G'_5(1\,1\,1\,1) = 0.125$$
$$G'_5(1\,1\,1\,0) = G'_5(0\,1\,1\,1) = -0.125$$

Clock correlation is then

$$\begin{aligned} \Delta C_5(m) = {} & 0.391 + 0.094\,C_2(m) + 0.31\,C_3(m) + 0.031\,C_4(m) \\ & + 0.094\,C_5(m) + 0.015\,C_2(m)\,C_3(m) + 0.015\,C_2(m)\,C_4(m) \\ & + 0.047\,C_2(m)\,C_5(m) - 0.015\,C_3(m)\,C_4(m) + 0.015\,C_3(m)\,C_5(m) \\ & + 0.015\,C_4(m)\,C_5(m) + -0.015\,C_2(m)\,C_3(m)\,C_4(m)\,C_5(m) \end{aligned}$$

Once these calculations are made, the logics given in Table A-1 are established. The decoding proceeds as follows (see Fig. A-1):

1. *First component:* clock-locks, 62.5% correlation.

2. *Second component:* with clock in lock, using g_2, correlation is 39.1% until the second component is acquired when correlation jumps 9.4% to 48.5%. If any of components 3, 4, or 5 are in lock, the initial clock-lock at this point may be 1.5% higher.

3. *Third component:* using g_3, assuming clock and second-component lock, the clock-lock correlation is 48.5% until third component acquisition, when correlation jumps 14.1% to 62.6%. Components 4 and 5 have no effect.

4. *Fourth component:* using g_4 and assuming components 2 and 3 are acquired, the clock correlation stands at 53.1% and jumps 15.6% to 68.7% as component 4 is locked. Component 5 does not affect the reading.

5. *Fifth component:* using g_5, assuming components 2, 3, and 4 are in lock, the clock-lock is initially 56.2%, jumping 15.6% to 71.8% as component 5 is stepped into phase.

6. *Final combination:* when all components are locked, the decoder logic is changed to f, and the correlation jumps to 100%.

Table A-1. Ranging encoding and decoding functions

x_1[a]	x_2	x_3	x_4	x_5	$\hat{g}_2$	$\hat{g}_3$	$\hat{g}_4$	$\hat{g}_5$	$\hat{f}$
0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0
0	0	0	1	1	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0
0	0	1	0	1	0	0	0	0	0
0	0	1	1	0	0	0	0	0	0
0	0	1	1	1	0	0	0	0	0
0	1	0	0	0	1	0	0	0	0
0	1	0	0	1	0	0	0	0	0
0	1	0	1	0	1	0	1	0	0
0	1	0	1	1	0	0	0	1	0
0	1	1	0	0	0	1	0	0	0
0	1	1	0	1	0	1	0	1	0
0	1	1	1	0	0	0	1	0	0
0	1	1	1	1	1	1	1	1	1
1	0	0	0	0	0	0	0	0	0
1	0	0	0	1	0	0	0	0	0
1	0	0	1	0	0	0	0	0	0
1	0	0	1	1	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0
1	0	1	0	1	0	0	0	0	0
1	0	1	1	0	0	0	0	0	0
1	0	1	1	1	0	0	0	0	1
1	1	0	0	0	0	0	0	0	0
1	1	0	0	1	1	0	0	0	0
1	1	0	1	0	0	0	0	0	0
1	1	0	1	1	0	0	1	1	1
1	1	1	0	0	1	1	0	0	0
1	1	1	0	1	0	0	0	1	1
1	1	1	1	0	0	1	1	0	1
1	1	1	1	1	1	1	1	1	1

[a] x_1 is the clock component.

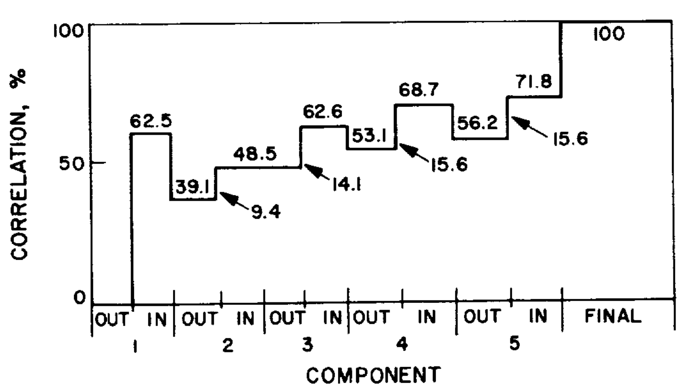

Fig. A-1. Acquisition diagram

APPENDIX B

Calculation of F(s) for Majority Logic

Let n be odd and let f be the unique majority logic. We consider only odd n, because if n were even

$$\frac{F_n(\mathbf{e}^1)}{F_{n+1}(\mathbf{e}^1)} = \frac{2^{1-n}\binom{n-1}{\frac{n}{2}}}{2^{-n}\binom{n}{\frac{n}{2}}} = 1$$

We could thus increase n by one without affecting the correlation value (or the correlation time) but decreasing the ratio $(\Sigma v_i)/(\Pi v_i)$.

We wish to calculate the transform of $f(\mathbf{x})$. Because f is a symmetric function, if $\mathbf{s}$ has k one's (i.e., $\mathbf{s} = k$), then for some permutation π

$$F(\mathbf{s}) = F(\pi\,\mathbf{u}^k) = F(\mathbf{u}^k)$$

$$\mathbf{u}^k = (1, 1, \cdots, 1, 0, \cdots, 0)$$

and by this symmetry of f, we need to calculate only these $F(\mathbf{u}^k)$.

$$\mathrm{F}(\mathbf{u}^k) = 2^{-n/2}\sum_{\mathbf{x}}(-1)^{x_1+\cdots+x_k} f(\mathbf{x})$$

$$= 2^{n/2}\left[\sum_{\mathbf{x},\,\|\mathbf{x}\| < \frac{n}{2}}(-1)^{x_1+\cdots+x_k} - \sum_{\mathbf{x},\,\|\mathbf{x}\| > \frac{n}{2}}(-1)^{x_1+\cdots+x_k}\right]$$

Define the two sums above as

$$A(k) = \sum_{\mathbf{x},\,\|\mathbf{x}\| < \frac{n}{2}} (-1)^{x_1+\cdots+x_k}$$

$$B(k) = \sum_{\mathbf{x},\,\|\mathbf{x}\| > \frac{n}{2}} (-1)^{x_1+\cdots+x_k}$$

Suppose that vector $\mathbf{x}$ has i one's in it, j of which lie in $x_1, \cdots, x_k$, and $i - j$ in $x_{k+1}, \cdots, x_n$. There are $\binom{k}{j}\binom{n-k}{i-j}$ such vectors $\mathbf{x}$, and thus

$$A(k) = \sum_{i=0}^{\frac{n-1}{2}} \sum_{j=0}^{\min(k,i)} \binom{k}{j}\binom{n-k}{i-j}(-1)^j$$

$$= \sum_{i=0}^{\frac{n-1}{2}} \sum_{j=0}^{\infty} \binom{k}{j}\binom{n-k}{i-j}(-1)^j$$

By similar reasoning,

$$B(k) = \sum_{i=\frac{n+1}{2}}^{n} \sum_{j=0}^{\infty} \binom{k}{j}\binom{n-k}{i-j}(-1)^j$$

Let $\mathcal{A}(t)$ be the generating function

$$\mathcal{A}(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{k}{j}\binom{n-k}{i-j}(-1)^j t^i$$

$$= \sum_{j=0}^{\infty}\binom{k}{j}(-1)^j t^j \sum_{i=0}^{\infty}\binom{n-k}{i-j}t^{i-j}$$

$$= \sum_{j=0}^{\infty}\binom{k}{j}(-1t)^j \sum_{m=0}^{\infty}\binom{n-k}{m}t^m$$

$$= (1-t)^k(1+t)^{n-k}$$

Note that the sum of the coefficients of $t^0, t^1, \cdots, t^{\frac{n-1}{2}}$ is precisely $A(k)$; that is,

$$A(k) = \text{coeff. of } t^{\frac{n-1}{2}} \text{ in } (1-t)^k(1+t)^{n-k}(1+t+\cdots + t^{\frac{n-1}{2}})$$

$$= \text{coeff. of } t^{\frac{n-1}{2}} \text{ in } (1-t)^{k-1}(1+t)^{n-k}(1-t^{\frac{n+1}{2}})$$

$$= \text{coeff. of } t^{\frac{n-1}{2}} \text{ in } (1-t)^{k-1}(1+t)^{n-k}$$

$$= \text{coeff. of } t^{n-1} \text{ in } (1-t^2)^{k-1}(1+t^2)^{n-k}$$

$$= \text{coeff. of } t^{-1} \text{ in } \frac{(1-t^2)^{k-1}(1+t^2)^{n-k}}{t^n}$$

By this procedure, we reduce $A(k)$ to the residue of a rational function, to be calculated by the Cauchy residue theorem:

$$A(k) = \frac{1}{2\pi j}\oint \frac{(1-t^2)^{k-1}(1+t^2)^{n-k}}{t^n}\,dt$$

$$(j = \sqrt{-1})$$

integrating along any simple closed path containing the origin.

$$A(k) = \frac{1}{2\pi j}\oint \left(\frac{1}{t} - t\right)^{k-1}\left(\frac{1}{t} + t\right)^{n-k}\frac{dt}{t}$$

Choose the integration path to be unit circle, $t = e^{jz}$.

$$A(k) = \frac{2^{n-2}(-j)^{k-1}}{\pi}\int_0^{2\pi} \sin^{k-1} z \cos^{n-k} z\,dz$$

Because $A(k)$ must be real, we may limit our attention to the real part of the equation (i.e., to odd k). This integral is one which can be reduced by a standard table of integrals [see Burington (Ref. 9), for example] to

$$A(k) = \mathcal{R}_e\left\{(-j)^{k-1}\frac{(k-1)!\left(\frac{n-k}{2}\right)!}{\left(\frac{n-1}{2}\right)!\left(\frac{k-1}{2}\right)!}\binom{n-k}{\frac{n-k}{2}}\right\}$$

By a similar procedure, or by invoking symmetry of the majority function, we compute

$$B(k) = A(k)$$

The final result for $F(\mathbf{s})$ is, then

$$F(\mathbf{u}^k) = 2^{1-n}\,\mathcal{R}_e\left\{(-j)^{k-1}\frac{(k-1)!\left(\frac{n-k}{2}\right)!}{\left(\frac{n-1}{2}\right)!\left(\frac{k-1}{2}\right)!}\binom{n-k}{\frac{n-k}{2}}\right\}$$

which, for $k = 1$, gives the result obtained previously for $F(\mathbf{e}^1)$:

$$F(\mathbf{e}^1) = 2^{1-n}\binom{n-1}{\frac{n-1}{2}}$$

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